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ON SOLVING THE PROBLEM OF AN IDEAL INCOMPRESSIBLE FLUID FLOW AROUND LARGE-ASPECT-RATIO AXISYMMETRIC BODIES

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Based on Babenko's fundamental mathematical ideas, principally new (unsaturated) algorithms are developed for the numerical solution of problems of a potential axisymmetric ideal fluid flow around bodies of revolution, in particular, an ellipsoid of revolution with an aspect ratio equal to 1000*.*

Key words: *flow problem, body of revolution, exterior Neumann problem, unsaturated numerical algorithm, exponential convergence.*

One of the few hydrodynamic problems considered in all reference books on hydrodynamics is the problem of a vortex-free flow of an ideal incompressible fluid around various bodies. Though the theory for these problems has been developed long ago and has acquired a canonical character (exterior Neumann problem for the Laplace equation), some important issues are still unresolved. Thus, up to now there remains a significant gap in solving the problem of the flow around three-dimensional large-aspect-ration bodies by numerical methods. Meanwhile, the development of science and engineering necessitates the numerical study of spatial fluid flows. Those sections of hydrodynamics where the exterior Neumann problem arises as an important intermediate stage are most difficult to analyze, and the correctness of studying the entire hydrodynamic problem depends on how carefully the exterior Neumann problem is solved numerically (e.g., in the case of problems described by boundary-layer equations [1]).

Situations where the exterior Neumann problem arises are so versatile that not all numerical methods are of interest for practice: a transition from a two-dimensional to a three-dimensional analysis increases not only the number of variables but also the volume of numerical information being processed and, hence, the time of operation of the numerical algorithm. There is the only possibility of avoiding these difficulties: using more perfect methods of discretization of problems considered [2]. As the theory of approximating functions on smooth manifolds homeomorphic to a two-dimensional sphere has not been developed yet, additional investigations are needed to overcome the difficulties of computer implementation of these problems for a three-dimensional body of an arbitrary shape. The problem can be solved rather easily only for the flow around axisymmetric bodies [3]. No precision numerical algorithms exist for large-aspect-ration axisymmetric bodies yet; therefore, the cycle of investigations [3–5] performed by the author and the numerical results obtained in these studies can be considered as pioneering in a certain sense.

One of the most important theoretical achievements of computational mathematics during the last 30 years is the development of principally new (unsaturated) numerical algorithms [2]. With increasing the supply of smoothness of the sought solutions, the computational process determined by an unsaturated algorithm is self-improving and reaches the peak of its efficiency (exponential convergence) on classes of problems with C^∞-smooth solutions [5]. As a result, the information on the extraordinary supply of smoothness, e.g., on infinite differentiability and analyticity, becomes fairly essential. Thus, numerical solutions of problems can be constructed extremely economically [4] (with exponential accuracy in the case of C^∞-smooth boundary data [3]).

Unsaturated numerical algorithms filling the gap in solving C^∞-smooth problems of an axisymmetric flow around large-aspect-ratio bodies of revolution are constructed in the present paper on the basis of Babenko's fundamental ideas [2].

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Without considering the capabilities of unsaturated algorithms in detail, we analyze the essential elements of their structure and pay the main attention to only one property of these algorithms. This property implies that the structure of unsaturated algorithms initially includes an infinite number of numerical methods.

Let us now formulate the problem and introduce the necessary definitions. Let ω be an axisymmetric domain in the space $\mathbb{R}^3$, which is bounded by a smooth closed surface of revolution $\partial\omega$ whose meridional section is a smooth curve γ: $[0,1] \to \{r(s), z(s)\}$ and $\gamma(s) \in C^\infty[0,1]$. Here $r = \sqrt{x^2+y^2}$; z are the invariants of the group of revolution of the domain ω with respect to the axis z.

The positions of the points $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ and $\boldsymbol{x} = (x, y, z)$ on $\partial\omega$ are defined by local coordinates (σ, ϕ) and (s, v), respectively,

$$\boldsymbol{\xi} = (\rho\cos\phi,\ \rho\sin\phi,\ \zeta), \qquad \boldsymbol{x} = (r\cos v,\ r\sin v,\ z),$$

$$\rho \equiv \rho(\sigma) = \sqrt{\xi^2+\eta^2}, \qquad \zeta \equiv \zeta(\sigma), \qquad r \equiv r(s) = \sqrt{x^2+y^2}, \qquad z \equiv z(s),$$

$$0 \leqslant \sigma, \qquad s \leqslant 1, \qquad 0 \leqslant \phi, \qquad v < 2\pi,$$

with orthonormalized bases corresponding to the point $\boldsymbol{\xi}, \boldsymbol{x} \in \partial\omega$:

$$\text{for} \quad \boldsymbol{\xi}, \qquad \boldsymbol{e} = \delta^{-1}\boldsymbol{\xi}_\sigma, \quad \boldsymbol{t} = \rho^{-1}\boldsymbol{\xi}_\phi, \quad \boldsymbol{n} = \boldsymbol{e}\times\boldsymbol{t};$$

$$\text{for} \quad \boldsymbol{x}, \qquad \boldsymbol{E} = \Delta^{-1}\boldsymbol{x}_s, \quad \boldsymbol{T} = r^{-1}\boldsymbol{x}_v, \quad \boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{T}.$$

Here $\boldsymbol{\xi}_\lambda$ and $\boldsymbol{x}_\lambda$ are the partial derivatives of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{x}$ with respect to the local coordinate λ; $\delta \equiv |\boldsymbol{\xi}_\sigma| = \sqrt{\rho_\sigma^2+\zeta_\sigma^2}$ and $\Delta \equiv |\boldsymbol{x}_s| = \sqrt{r_s^2+z_s^2}$.

The surface in the space $\mathbb{R}^3$ is understood as a closed bounded surface of revolution $\partial\omega \in C^\infty$.

The mapping $\partial F/\partial\boldsymbol{x}$: $\mathbb{R}^3 \to \mathbb{R}^3$ acting on the vectors $\boldsymbol{c} \in \mathbb{R}^3$ as a linear form $(\partial F/\partial\boldsymbol{x})\langle\boldsymbol{c}\rangle = \nabla_{\boldsymbol{x}}F\langle\boldsymbol{c}\rangle$ is the gradient of the scalar function F: $\mathbb{R}^3 \to \mathbb{R}$. Let us identify it with the covector $\nabla_{\boldsymbol{x}}F$. The direct value of the function $F(\boldsymbol{x})$ on the surface $\partial\omega$ (provided that it exists) is determined by the equality obtained by replacing the point $\boldsymbol{x} \in \mathbb{R}^3$ by the point $\boldsymbol{x} \in \partial\omega$, i.e., we assume that $\bar{F}(\boldsymbol{x}) = F(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\omega}$. For the direct and limiting values (inside and outside the surface $\partial\omega$) of the derivative $\partial F/\partial\boldsymbol{A} = \nabla_{\boldsymbol{x}}F\langle\boldsymbol{A}\rangle$ along the direction of the vector $\boldsymbol{A}$, we use the notation $(A\bar{F})(\boldsymbol{x})$, $(A_+\bar{F})(\boldsymbol{x})$, $(A_-\bar{F})(\boldsymbol{x})$, and $\boldsymbol{x} \in \partial\omega$; the values of the vector field $\boldsymbol{A}$ at the points $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are indicated by the capital and lower-case Latin letters $\boldsymbol{A} \equiv \boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{a} \equiv \boldsymbol{A}(\boldsymbol{\xi})$.

The problem of the vortex-free flow around a body of revolution ω with the normal $\boldsymbol{N}$ at the point $\boldsymbol{x} \in \partial\omega$ by an ideal incompressible fluid reduces to finding the velocity potential φ, which is a harmonic function outside the body ω. The potential φ satisfies the boundary conditions $N_-\bar{\varphi}\Big|_{\partial\omega} = 0$ on the body and $\boldsymbol{u} = \nabla\varphi \to \boldsymbol{U}_\infty$ at infinity [1]. Without loss of generality, we assume that $\boldsymbol{U}_\infty \equiv (0, 0, U)$. The fluid flow considered is assumed to be axisymmetric. Let $\Phi = \varphi - Uz$. Then, we obtain

$$\Delta\Phi = 0, \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \omega; \qquad N_-\bar{\Phi}\Big|_{\partial\omega} = -U\cos(N, z); \qquad \Phi \to 0 \quad \text{for} \quad |\boldsymbol{x}| \to \infty. \tag{1}$$

A decrease in Φ at infinity ensures the uniqueness of the solution of problem (1). Solving the problem of the flow around a body also implies calculating the tangent in the direction $\boldsymbol{E}$ of velocity of the fluid on the body surface, i.e., the tangent at the point $\boldsymbol{x} \in \partial\omega$ of the boundary gradient of the solution $\nabla\varphi\langle\boldsymbol{E}\rangle\Big|_{\partial\omega} = E_-\bar{\varphi}$ of problem (1), and the pressure p. The pressure p (which is dimensionless) is calculated from the Bernoulli integral by the formula $p = \{1 - [\varphi_r^2 + (1-\varphi_z)^2]\}/2 + p_\infty$, where p_∞ is the pressure at infinity.

Thus, the solution of problem (1) is a particular case of the solution of the exterior Neumann problem

$$\Delta\Phi = 0 \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \omega; \qquad N_-\bar{\Phi}\Big|_{\partial\omega} = f(\boldsymbol{x}); \qquad \Phi \to 0 \quad \text{for} \quad |\boldsymbol{x}| \to \infty. \tag{2}$$

The function $f(\boldsymbol{x})$ in the cylindrical coordinates (r, z, v) is independent of v. Hence, we can confine ourselves to the solutions $\Phi(\boldsymbol{x})$ of problem (2), which depend only on the invariants r and z. As the function $f(s) \equiv f(r(s), z(s))$ is further assumed to be sufficiently smooth along the arc coordinate $s \in [0,1]$, the function f is continued in an even continuous manner with an unchanged class of smoothness to the segment $[1,2]$, thus, becoming a continuous periodic function with the period 2 (2-periodic function). By virtue of Schauder's estimates, the solutions of problem (2) becomes as smooth as it is permitted by the function $f(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\omega$.

We introduce a class $C[0,2]$ of continuous 2-periodic functions and denote the norm by $\|\cdot\|$. The set of even 2-periodic functions forms a closed subspace in $C[0,2]$. Let us denote the latter by $C_+ \equiv C_+[0,1]$.

The numerical solution of the exterior Neumann problem (2) is based on the methods of the potential theory [6]: the boundary integral equation equivalent to problem (2) is a fairly efficient tool for the numerical solution of this problem.

Let us consider a harmonic potential of a simple layer $V[\psi](\boldsymbol{x})$ with a continuous density $\psi(\boldsymbol{\xi}) \in C(\partial\omega)$ invariant with respect to revolutions $\partial\omega$:

$$V[\psi](\boldsymbol{x}) = \int\limits_{\partial\omega} \frac{\psi(\boldsymbol{\xi})}{|\boldsymbol{\xi}-\boldsymbol{x}|}\, d\omega_{\boldsymbol{\xi}}.$$

Here $|\boldsymbol{\xi}-\boldsymbol{x}|$ is the distance between the points $\boldsymbol{\xi} \in \partial\omega$ and $\boldsymbol{x} \in \mathbb{R}^3$.

If $\psi(\boldsymbol{\xi}) \in C^{1+\alpha}(\partial\omega)$ $(0<\alpha<1)$, the potential $V[\psi](\boldsymbol{x})$ has the following properties [6]:

1) Everywhere outside $\partial\omega$, the potential $V[\psi](\boldsymbol{x})$ satisfies the Laplace equation and determines the harmonic functions $V^+(\boldsymbol{x})$ and $V^-(\boldsymbol{x})$ inside and outside $\partial\omega$, respectively;

2) The potential $V[\psi](\boldsymbol{x})$ is continuous everywhere in $\mathbb{R}^3$, and $V^+(\boldsymbol{x}) = \bar{V}(\boldsymbol{x}) = V^-(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial\omega$;

3) The limits $N_\pm \bar{V}[\psi](\boldsymbol{x})$ on $\partial\omega$ exist and are continuous functions

$$N_\pm \bar{V}[\psi](\boldsymbol{x}) = \pm 2\pi\psi(\boldsymbol{x}) + N\bar{V}[\psi](\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \partial\omega,$$

and the operator $N\bar{V}$ has a weak singularity on $\partial\omega$;

4) The limits $E_\pm \bar{V}[\psi](\boldsymbol{x})$ on $\partial\omega$ exist and are continuous functions with

$$E_+\bar{V}[\psi](\boldsymbol{x}) = E_-\bar{V}[\psi](\boldsymbol{x}) = E\bar{V}[\psi](\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \partial\omega.$$

By virtue of property 3, seeking for the solution of problem (2) in the form $\Phi(\boldsymbol{x}) = V[\psi](\boldsymbol{x})$ is equivalent to solving the boundary integral equation

$$-2\pi\psi(s) + N\bar{V}[\psi](s) = f(s) \qquad [s \in [0,1] \quad (\psi, f \in C_+)] \tag{3}$$

with the operator $N\bar{V}$ being compact in C_+.

Let us indicate the presentations of the operators $N\bar{V}[\psi](\boldsymbol{x})$, $\bar{V}[\psi](\boldsymbol{x})$, and $E\bar{V}[\psi](\boldsymbol{x})$ arising in solving problem (1), which are convenient from the viewpoint of numerical implementation.

We consider the complete elliptic integrals with the modulus α

$$K(\alpha) = \int\limits_0^{\pi/2} (1-\alpha\sin^2\theta)^{-1/2}\, d\theta, \qquad E(\alpha) = \int\limits_0^{\pi/2} (1-\alpha\sin^2\theta)^{1/2}\, d\theta, \qquad D(\alpha) = K(\alpha) - E(\alpha)$$

and introduce the following notation:

$$h_* \equiv h_*(\sigma,s) = \sqrt{(\rho+r)^2 + (\zeta-z)^2}, \qquad q \equiv q(\sigma,s) = 4\rho r h_*^{-2}.$$

The expression $|\boldsymbol{\xi}-\boldsymbol{x}| = h_*[1-q\cos((\phi-v)/2)]^{1/2}$ is valid for $\boldsymbol{\xi}, \boldsymbol{x} \in \partial\omega$, and the direct values of the above-indicated integral operators have the following form:

$$N\bar{V}[\psi](s) = 2r^{-1}\int\limits_0^1 \psi(\sigma)\Big(2\rho r\,\frac{\boldsymbol{H}\cdot\boldsymbol{N}}{\sigma-s}\,E(q) + \rho\Delta^{-1}z_s D(q)\Big)h_*^{-1}\delta\, d\sigma; \tag{4}$$

$$\bar{V}[\psi](s) = 2r^{-1}\int\limits_0^1 [2\rho r\psi(\sigma)]K(q)h_*^{-1}\delta\, d\sigma; \tag{5}$$

$$E\bar{V}[\psi](s) = 2r^{-1}\int\limits_0^1 2\rho r\,\frac{\psi(\sigma)\,\boldsymbol{H}\cdot\boldsymbol{E} - \psi(s)\,\boldsymbol{H}\cdot\boldsymbol{e}}{\sigma-s}\,E(q)h_*^{-1}\delta\, d\sigma$$

$$-\,2r^{-1}\int\limits_0^1 [\rho r_s\Delta^{-1}\psi(\sigma) + r\rho_\sigma\delta^{-1}\psi(s)]D(q)h_*^{-1}\delta\, d\sigma + 2r^{-1}\int\limits_0^1 [2\psi(s)r\rho_\sigma\delta^{-1}]K(q)h_*^{-1}\delta\, d\sigma. \tag{6}$$

Here the vector $\boldsymbol{H}(\sigma,s)$ has the form

$$\boldsymbol{H}(\sigma,s)=\frac{\boldsymbol{r}(\sigma,s)}{|\boldsymbol{r}(\sigma,s)|^2},\qquad \boldsymbol{r}(\sigma,s)=\Big[\frac{\rho-r}{\sigma-s},\ \frac{\zeta-z}{\sigma-s}\Big].$$

Presentations (4)–(6) have the following generic notation:

$$2r^{-1}\int\limits_0^1 F(\sigma,s)\Psi(q)h_*^{-1}\delta\,d\sigma.$$

Here $F(\sigma,s)$ is a function uniformly continuous in the domain $[0,1]\times[0,1]$ and $\Psi(q)$ is the complete elliptic integral $K(q)$, $E(q)$, or $D(q)$ with the modulus $q\equiv q(\sigma,s)$.

As q is a symmetric function of two variables, special requirements are imposed on the method of calculating the complete elliptic integral $\Psi(q)$.

Theorem 1. *For an arbitrary integer $p\geqslant 0$, the following expansion is valid:*

$$\Psi(q)=-\psi_p^*(q)\ln(1-q)+\Psi_p^*(q).$$

The functions $\psi_p^*(q)$ and $\Psi_p^*(q)$ and the methods for calculating these functions are described in [7].

With allowance for this expansion, equalities (4)–(6) are written as

$$2r^{-1}\int\limits_0^1 F(\sigma,s)\Psi_p^*(q)h_*^{-1}\delta\,d\sigma-2r^{-1}\int\limits_0^1 F(\sigma,s)\psi_p^*(q)\ln(1-q)h_*^{-1}\delta\,d\sigma. \tag{7}$$

Note that presentation (7) does not contain a logarithmic singularity in the poles $\gamma(0)$ and $\gamma(1)$ of the surface of revolution $\partial\omega$.

The complicated structure of presentations (4)–(6) points to the character of computational difficulties in axisymmetric problems as compared, for instance, with a planar case. By virtue of (7), the integrands in Eqs. (4)–(6), on one hand, have a “moving” logarithmic singularity on the diagonal $\sigma=s$ and, on the other hand, have zones of intense growth (boundary layers) at the points s located near the axis of symmetry of the surface $\partial\omega$ [5]. The widespread underestimation of the boundary-layer effect on the course of computations is one of the main drawbacks of the existing numerical techniques for solving axisymmetric problems. The commonly accepted identification of the moving logarithmic singularity by the principle $\ln(1-q)=2\ln|\sigma-s|+A(\sigma,s)$ $[\sigma\in(0,1),\ s\in(0,1)]$ turned out to be inapplicable near the axis of symmetry because the function $A(\sigma,s)$ is not uniformly continuous in the domain $[0,1]\times[0,1]$. Therefore, if any saturated quadrature formulas are used to approximate presentations (4)–(6), the loss of accuracy increases with decreasing distance between the point s and the axis of symmetry of the surface $\partial\omega$. The boundary layer is an integral part of all axisymmetric problems, and standard numerical methods fail to overcome this basic computational difficulty. Meanwhile, computational difficulties of this kind are quite natural for axisymmetric problems because they are caused by cylindrical symmetry. Formally, they are related to the presence of the weight factor $h_*^{-1}(\sigma,s)$ in Eq. (7), which turned out to be a universal characteristic of growth of integrands in (4)–(6) at points close to the axis of symmetry. The presence of the boundary layer in these problems was previously ignored. At the same time, it affects the numerical implementation of axisymmetric problems. To obtain nontrivial numerical results, one has to solve a number of mathematical problems. In particular, to avoid loss of accuracy caused by the moving logarithmic singularity, this singularity had to be taken into account more carefully [3], and a principally novel approach was developed to cancel the effects induced by the boundary layer [5].

Let us transform Eq. (7) to the form that allows using the results of [3, 5]. We introduce the following notation:

$$R_*^2\equiv R_*^2(\sigma,s)=\Big(\frac{\rho+r}{\sin(\pi(\sigma+s)/2)}\Big)^2+\Big(\frac{\zeta-z}{\sin(\pi(\sigma+s)/2)}\Big)^2,$$

$$R^2\equiv R^2(\sigma,s)=\Big(\frac{\rho-r}{\sin(\pi(\sigma-s)/2)}\Big)^2+\Big(\frac{\zeta-z}{\sin(\pi(\sigma-s)/2)}\Big)^2,$$

$$B\equiv B(\sigma,s)=\frac{R^2(\sigma,s)}{R_*^2(\sigma,s)},\qquad Q(\sigma,s)=\frac{4(r/\sin\pi s)(\rho/\sin\pi\sigma)}{R_*^2(\sigma,s)},$$

$$b\equiv b(\sigma,s)=\ln B(\sigma,s),\qquad a\equiv a(\sigma,s)=\delta R_*^{-1}(\sigma,s)\sin(\pi(\sigma+s)/2).$$

In this notation, the modulus of elliptic integrals q and the weight factor h_*^{-1} have the form

$$q(\sigma, s) = \frac{\sin \pi\sigma \sin \pi s}{\sin^2(\pi(\sigma + s)/2)}\, Q(\sigma, s), \qquad h_*^{-1}(\sigma, s) = \frac{R_*^{-1}(\sigma, s)}{\sin(\pi(\sigma + s)/2)}.$$

The value of $s \in (0, 1)$ being fixed, we perform the following implicit replacement of the integration variable σ in Eq. (7):

$$t \equiv t(\sigma, s) = \frac{\sin(\pi(\sigma - s)/2)}{\sin(\pi(\sigma + s)/2)}, \qquad \sigma \in [0, 1]. \tag{8}$$

The mapping $t\colon [0, 1] \to [-1, 1]$ reveals the structure of the integrand functions in Eq. (7):

$$\tilde{q} = (1 - t)(1 + t)\tilde{Q}(t), \qquad 1 - \tilde{q} = t^2 \tilde{B}(t), \qquad h_*^{-1}\delta\, d\sigma = \varepsilon^{-1}\tilde{a}(t)\, dt.$$

In this case, the "moving" logarithmic singularity transforms to a motionless singularity, which is the middle of the segment; by virtue of the equality

$$\Big(\frac{d}{dt}\Big)^k \tilde{g}(t) = \varepsilon^{-k}\Big(\sin^2\frac{\pi(\sigma + s)}{2}\,\frac{d}{d\sigma}\Big)^k g(\sigma, s) \qquad \Big(\varepsilon = \frac{\pi}{2}\sin \pi s \quad \forall\ k \geqslant 0\Big), \tag{9}$$

the boundary layer of thickness ε is explicitly captured [5]. The operation "$\sim$" acts on all functions $g(\sigma, s)$ uniformly continuous in the domain $[0, 1] \times [0, 1]$ as follows: $\tilde{g} \equiv \tilde{g}(t) = g(\sigma(t, s), s)$, where the function $\sigma(t, s)$ for a fixed value of $s \in (0, 1)$ is inverse to the function $t(\sigma, s)$.

Thus, by replacing the variable of integration in Eq. (8), we transform expression (7) to

$$2\varepsilon^{-1}r^{-1}\int_{-1}^{1}\tilde{F}_c(t)\tilde{a}(t)\, dt - 2\varepsilon^{-1}r^{-1}\int_{-1}^{1}\tilde{F}_d(t)\tilde{a}(t)\ln|t|\, dt.$$

The approximate numerical implementation of this integral expression and relations (4)–(6) is performed with the use of the quadrature formula

$$2\varepsilon^{-1}r^{-1}\sum_{k=1}^{n}\Big[c_k\tilde{F}_c(t_k) + d_k\tilde{F}_d(t_k)\Big]\tilde{a}(t_k), \qquad \tilde{a}(t_k) = a(\sigma(t_k, s), s). \tag{10}$$

Here $t_k = \cos(\pi(2k - 1)/(2n))$ are the nodes; $c_k > 0$ and $d_k > 0$ ($k = 1, 2, \ldots, n$) are the weight factors of unsaturated quadrature formulas [3]. The parameter $p \geqslant 0$ present in algorithms for computing the complete elliptic integrals $\Psi(q)$ (see Theorem 1) is responsible for smoothness of functions in the integrands in Eqs. (4)–(6): the functions belong to the class $C^{2p+1}[-1, 1]$. The parameter p is chosen in accordance with the condition of neutralization in presentations (4)–(6) of the boundary layer of thickness $\varepsilon = 0.5\pi\sin \pi s$, i.e., $n > n_{\min}$ and $p \geqslant 10$ [5]. The values of $\sigma(t_i, s) \equiv \sigma_i$ ($1 \leqslant i \leqslant n$) in formulas (10) are found from the equations

$$t_i = \frac{\sin(\pi(\sigma_i - s)/2)}{\sin(\pi(\sigma_i + s)/2)} \qquad [i = 1, \ldots, n, \quad s \in (0, 1)]$$

with the use of Newton's method:

$$y^{(\alpha+1)} = y^{(\alpha)} - f(y^{(\alpha)})/f'(y^{(\alpha)}), \qquad \alpha = 0, 1, 2, \ldots\ ,$$

$$f(x) = x - s - 2\arcsin\Big(\frac{t_i\sin(\pi(x + s)/2)}{2}\Big), \quad f'(x) = 1 - t_i\,\frac{\cos(\pi(x + s)/2)}{\sqrt{1 - t_i^2\sin^2(\pi(x + s)/2)}}.$$

The initial approximation $y^{(0)} = \sigma_i^{(0)}$ ($1 \leqslant i \leqslant n$) is chosen as follows:

$$\sigma_1^{(0)} = 1 - (1 - t_1)\cot(\pi s/2)/\pi,$$

$$\sigma_{i+1}^{(0)} = \sigma_i + 2\pi^{-1}(t_{i+1} - t_i)\,\frac{\sin^2(\pi(\sigma_i + s)/2)}{\sin \pi s}, \qquad 1 \leqslant i \leqslant n - 1.$$

This iterative process converges quadratically; therefore, for $s \in [0.001, 0.999]$, two or three iterations already ensure at least ten correct decimal digits for the values of σ_i ($1 \leqslant i \leqslant n$). The values of σ_i ($1 \leqslant i \leqslant n$) are automatically constructed in numerical implementation of formulas (10) [5], depending on the boundary-layer thickness $\varepsilon = 0.5\pi \sin \pi s$.

Thus, under the condition that the mapping t: $[0, 1] \to [-1, 1]$, which brings the boundary integral operators (4)–(6) to the form (7), is found only by virtue of additional smoothness of problems (1) and (2), it is possible (see [3]), first, to take into account the geometry of meridional cross sections of C^∞-smooth axisymmetric domains and, second, to reduce neutralization of the boundary layer identified explicitly by relations (9) to using appropriate properties of unsaturated quadrature formulas [5], which take into account the specific behavior of the integrands on the diagonal $s = \sigma$. The properties of well-posedness of the unsaturated quadratures (10) used to approximate Eq. (7) ensure robustness of computational processes with respect to rounding errors [3].

In solving problems (1) and (2), we use the equivalent boundary integral equation (3), confining ourselves to considering C^∞-smooth axisymmetric domains and rather smooth axisymmetric solutions. Let us give an informal description of a finite-dimensional unsaturated approximation of Eq. (3). First, it is necessary to choose a suitable unsaturated method of approximation of the solution ψ proper. We use $e_m(g)$ to denote the best Chebyshev approximation of the continuous periodic function $g \in C[0, 2]$ by trigonometric polynomials of the order lower than or equal to m. The higher the smoothness of the function g, the better the accuracy $e_m(g)$ of approximation of the function g in $C[0, 2]$ by a trigonometric polynomial of the best Chebyshev approximation. Moreover, characterization of 2-periodic functions of finite smoothness is performed asymptotically with $e_m(g)$ decreasing to zero as $m \to \infty$ [with the use of direct and inverse Jackson's theorems, which establish correspondence between the properties of smoothness of the function $g \in C[0, 2]$ and statements of the form $m^\varsigma e_m(g) \to 0$ (the value of $\varsigma \geqslant 0$ is finite)]. It is this property (being a structural carrier of information on the differential nature of $g \in C[0, 2]$) that determines a special status of polynomials of the best Chebyshev approximation. Note that the specific features of the behavior of characteristics of $e_m(g)$ with increasing parameter $m \geqslant 0$ for classes of 2-periodic C^∞-smooth functions was considered in [5]. It follows from the results of [5] that the method of approximation of 2-periodic functions by polynomials of the best Chebyshev approximation does not possess the saturation property. This circumstance is used below.

Let $s_i = 2i/(2m + 1)$, $0 \leqslant i \leqslant m$. The mapping J: $C_+ \to \mathbb{R}^{m+1}$ determined by the equality $Jg = (g(s_0), \ldots, g(s_m))$ has a decoding algorithm based on calculating the interpolation Lagrange polynomial

$$(Q_m g)(s) \equiv Q_m(s; Jg) = \frac{2}{2m+1} \sum_{k=0}^{m} g(s_k) v_k(s),$$

where

$$v_k(s) = \begin{cases} D_m(\pi s), & k = 0, \\ D_m(\pi(s - s_k)) + D_m(\pi(s + s_k)), & k = 1, 2, \ldots, m, \end{cases}$$

$D_m(s) = 1/2 + \sum\limits_{k=1}^{m} \cos ks$ is the Dirichlet kernel. It follows from the Lebesgue inequality

$$\|g(s) - Q_m(s; Jg)\| \leqslant (1 + \|Q_m\|)\, e_m(g)$$

and the results of [5] that the above-mentioned method of approximation of the function $g \in C_+^\infty[0, 1]$ simultaneously with its derivatives does not possess the property of saturation, and $\|Q_m\| \leqslant 3 + 4\pi^{-2} \ln m$ (see [2]).

The discrete analog of Eq. (3) is obtained as follows. Let

$$\psi(s) = Q_m(s; J\psi) + \rho_m(s; \psi), \quad u = J\psi, \quad \nu = -Jf, \quad \vartheta = JN\bar{V}[\rho_m].$$

Then, Eq. (3) yields the relation $u + Au = \nu + \vartheta$, where $A = (a_{ij})$ is a matrix of size $(m+1) \times (m+1)$ with elements $a_{ik} \equiv a_k(s_i) = -N\bar{V}[v_k](s_i)$ ($0 \leqslant k \leqslant m$ and $0 \leqslant i \leqslant m$). The value of a_{ik} is approximately calculated with the use of unsaturated quadratures (10). The use of the latter with the number of nodes $n > n_{\min} > m \geqslant m_0$ allows us to calculate the matrix A with accuracy of $(1 + 2\|Q_m\|) e_m(\psi)$ (see [3]). The numerical implementation $N\bar{V}[g]$ is performed on the basis of representation (4) if formulas (10) have

$$\tilde{F}_c(t) = F_c(\sigma(t, s), s), \qquad \tilde{F}_d(t) = F_d(\sigma(t, s), s),$$

$$F_c(\sigma, s) = g[\Omega(E^* - e^* b) + \omega(D^* - d^* b)], \qquad F_d(\sigma, s) = 2g[\Omega e^* + \omega d^*],$$

$$\Omega(\sigma, s) = 2\rho r(\sigma - s)^{-1} \boldsymbol{H}(\sigma, s) \cdot \boldsymbol{N}, \qquad \omega(\sigma, s) = \rho z_s \Delta^{-1}, \qquad \varepsilon = 0.5\pi \sin \pi s.$$

The algorithms of calculating the functions E^*, e^*, D^*, and d^* are described in [7]. The common argument of these functions is the modulus of the elliptic integrals $\tilde{q} \equiv \tilde{q}(t) = q(\sigma(t,s),s)$.

If we neglect the approximation error $\vartheta \in \mathbb{R}^{m+1}$ and denote the approximate value of $J\psi \in \mathbb{R}^{m+1}$ by $\bar{\psi} \in \mathbb{R}^{m+1}$, we obtain the sought discretization of problem (3):

$$(I+A)\bar{\psi} = \nu \tag{11}$$

(I is the unit matrix). The Chebyshev norms of the vectors $Jg \in \mathbb{R}^{m+1}$ and the correlated norms of the matrices B: $\mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ are denoted by $|Jg| = \max\limits_{0\leqslant k\leqslant m} |g(s_k)|$ and $|B|$, respectively; the measure of conditionality of the matrix B is assumed to be the number $\varkappa(B) = |B|\,|B^{-1}|$.

Let us formulate the results obtained.

Let problem (3) posed in a C^∞-smooth axisymmetric domain ω be solvable for all right sides of $f \in C_+$ and $\|\psi\| \leqslant M\|f\|$. Then, the following statements are valid [3].

Theorem 2. *If $f \in C_+^l[0,1]$ and $l > 0$ is a sufficiently large integer, then Eq.* (11) *with $n > n_{\min} > m \geqslant m_0$ is solvable for all right sides of ν. In this case, $\|N\bar{V}[Q_m]\| \leqslant q_0 < \infty$, $\varkappa(I+A) \leqslant \varkappa_0 < \infty$, and*

$$e_m(\psi) \leqslant |J\psi - \bar{\psi}| \leqslant c_0(1+\|Q_m\|)e_m(\psi),$$

$$e_m(\psi) \leqslant \|\psi(s) - Q_m(s;\bar{\psi})\| \leqslant c_0\|Q_m\|(1+\|Q_m\|)e_m(\psi).$$

The constants q_0, $\varkappa_0$, and c_0 are independent of the parameter m and can be chosen as follows:

$$q_0 = \sup_{m\geqslant 0} \|N\bar{V}[Q_m]\|, \qquad \varkappa_0 = 2(1+Mq_0)^2, \quad c_0 = 2(1+Mq_0)\,\|K\|.$$

Theorem 3. *System* (11) *can be solved by an iterative method following the scheme*

$$\bar{\psi}^{k+1} = (1-\beta)\bar{\psi}^k - \beta A\bar{\psi}^k + \beta\nu, \qquad \beta = (1+\|N\bar{V}[Q_m]\|)^{-1}, \quad k = 0,1,\dots\ .$$

The iterations converge with the rate of a geometric progression with a denominator $\tau < 1$.

Theorem 4. *If $|(I+A)\bar{\psi}^k - \nu|/|\nu| \leqslant \epsilon/\varkappa(I+A)$, then $|\bar{\psi}^k - \bar{\psi}|/|\bar{\psi}| \leqslant \epsilon$.*

Remark 1. In contrast to methods that have the main term of the error (e.g., finite-difference methods), the numerical methods constructed here have an important advantage: they are unsaturated because the right sides of inequalities in Theorem 2 with increasing parameter m automatically track the differential properties of the exact solution $\psi(s) \in C_+[0,1]$ of problem (3), tuning to optimal estimates of the error, based on the actual smoothness of the exact solution. For $\psi \in C_+^\infty[0,1]$, the error decreases exponentially with increasing parameter m, and the required accuracy is reached with moderate values of m [5].

As an example, we considered problem (3) in a domain bounded by an ellipsoid of revolution with semiaxes $a > 0$ and $b > 0$. This domain is characterized by the only numerical parameter a/b. By decreasing or increasing this parameter, we can control the range of computational difficulties of the problem considered. In the computations performed, the parameter a/b was chosen such that the advantages of the new (unsaturated) technique for the numerical solution of C^∞-smooth (elliptic) problems (1) and (2) could be clearly demonstrated. Loitsyanskii [8] considered the problem for smooth bodies of revolution with a large aspect ratio ($b \gg a$). Until recently, it has been assumed that this problem cannot be solved numerically because saturated (with the main error term) numerical methods were used. The aspect ratios 1/10 and 10/1 are already critical for numerical techniques based on finite-difference or finite-element approximations or in other similar cases [2].

Vice versa, unsaturated numerical methods constructed by the author of the present paper are characterized by the absence of the main error term and, hence, can automatically tune to all values of robust smoothness of the solutions. Thus, the limit inaccessible for other numerical methods is overcome. This ensures high accuracy of numerical solutions designed with a small number of node points. Despite the extreme shapes of the domains chosen for test computations [“spike” ($a/b = 1/1000$) and “disk” ($a/b = 100/1$)], we managed to find precision numerical solutions with a limited amount of numerical information processed.

Design of the computational algorithm was aimed at obtaining a large number of correct decimal digits in the numerical solution being sought with a comparatively small dimensions of the matrices of linear systems to which the elliptic problem (3) reduces. As a result, solutions with 5 to 8 correct decimal digits were obtained for the following values of parameters of Theorems 1–4: $p = 10$, $m = 20$, $n = 501$, and $k = 10$.

Before giving the computation results, we indicate the exact solutions of problem (3) in ellipsoids of revolution extended ($b > a$) and flattened ($b < a$) along the axis of symmetry z [7]. The boundary data are the traces of the derivatives of harmonic polynomials.

We consider orthogonal curvilinear coordinate systems of the extended and flattened ellipsoids of revolution. These coordinates are related to the Cartesian rectangular coordinates (r, z) of the meridional section of the axisymmetric domain by the expressions

$$r = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}, \qquad z = c\lambda\mu, \qquad -1 \leqslant \mu \leqslant 1, \quad 1 \leqslant \lambda < \infty,$$

$$r = c\sqrt{(\lambda^2 + 1)(1 - \mu^2)}, \qquad z = c\lambda\mu, \qquad -1 \leqslant \mu \leqslant 1, \quad 0 \leqslant \lambda < \infty,$$

where c is a certain scale factor.

Let $\lambda = \lambda_0$ be the equation of the surface of the ellipsoid of revolution defined by the equation

$$r = c\sqrt{(\lambda_0^2 \mp 1)(1 - \mu^2)} = a \sin \pi s, \qquad z = c\lambda_0\mu = b\cos \pi s, \qquad 0 \leqslant s \leqslant 1$$

($a > 0$ and $b > 0$ are constants). Differentiation at the point $\boldsymbol{x} \in \partial\omega$ along the normal direction $\boldsymbol{N}$ and tangential direction $\boldsymbol{E}$ to this surface is performed by the operators

$$N \equiv \pi\Delta^{-1}(\lambda_0^2 \mp 1)^{1/2}\, \partial/\partial\lambda, \qquad E \equiv -\Delta^{-1}\, \partial/\partial s,$$

where

$$\lambda_0 = b(\pm b^2 \mp a^2)^{-1/2}, \qquad \Delta = \pi b(1 \mp \lambda_0^{-2}\mu^2)^{1/2}, \qquad \mu = \cos \pi s$$

[hereinafter, the upper and lower signs refer to the ellipsoid of revolution extended ($b > a$) and flattened ($b < a$) along the axis of symmetry z].

The solution of the exterior Neumann problem (2) with the right side

$$f(s) = 2\pi ab(\sin^2 \pi s - 2\cos^2 \pi s)/\Delta, \qquad \boldsymbol{x} \in \partial\omega \tag{12}$$

is sought in the form $\Phi(\lambda, \mu) = V[\psi](\lambda, \mu)$. As a result, we obtain the equalities

$$\psi(\mu) = -ab\Delta^{-1}\Lambda_2(3\mu^2 - 1)/2, \quad \bar{\Phi}(\lambda_0, \mu) = -2c^2(1 + \Lambda_2)p(\lambda_0)(3\mu^2 - 1)/3, \tag{13}$$

where

$$c = b/\lambda_0, \qquad p(\lambda_0) = ({3\lambda_0}^2 \mp 1)/2, \qquad \Lambda_2^{-1} = 3\lambda_0(\lambda_0^2 \mp 1)Q_2(\lambda_0) - 1,$$

$$Q_2(\lambda_0) = \begin{cases} [({3\lambda_0}^2 - 1)\ln((\lambda_0 + 1)/(\lambda_0 - 1)) - 6\lambda_0]/4, & b > a, \\ [({3\lambda_0}^2 + 1)\arcsin({\lambda_0}^2 + 1)^{-1/2} - 3\lambda_0]/2, & b < a. \end{cases}$$

TABLE 1

Neumann Problem ($n = 501$; $m = 20$)

j	$a=1$; $b=100$		$a=100$; $b=1$	
	$\bar{\Phi}$	Exact value of $\bar{\Phi}$	$\bar{\Phi}$	Exact value of $\bar{\Phi}$
0	15.2160167	15.2164331	156.7800796	156.7815430
1	14.6846869	14.6845775	151.2999018	151.3016038
2	13.1387293	13.1385838	135.3730077	135.3725562
3	10.7227032	10.7225496	110.4785577	110.4791023
4	7.6618324	7.6616667	78.9412912	78.9414916
5	4.2414083	4.2412315	43.6992966	43.6992570
6	0.7803051	0.7800533	8.0369847	8.0372292
7	−2.3987682	−2.3992612	−24.7208415	−24.7206337
8	−4.9994474	−5.0003768	−51.5208466	−51.5210623
9	−6.7794422	−6.7808510	−69.8666100	−69.8660633
10	−7.5730486	−7.5747305	−78.0454585	−78.0457503
11	−7.3064346	−7.3080202	−75.2980240	−75.2977175
12	−6.0043999	−6.0055793	−61.8782300	−61.8781012
13	−3.7881124	−3.7888047	−39.0376591	−39.0377067
14	−0.8639691	−0.8643159	−8.9056654	−8.9054233
15	2.4955044	2.4953038	25.7099708	25.7102025
16	5.9770838	5.9769134	61.5829699	61.5827446
17	9.2561643	9.2560020	95.3678078	95.3686234
18	12.0270887	12.0269347	123.9190917	123.9187512
19	14.0315860	14.0314407	144.5713245	144.5720499
20	15.0828305	15.0826854	155.4016491	155.4034831

TABLE 2

Flow Problem ($m = 20$)

j	$a=1$; $b=3$; $n=71$		$a=3$; $b=1$; $n=71$		$a=0.5$; $b=500$; $n=501$	
	$E\bar{\varphi}$	Exact value of $E\bar{\varphi}$	$E\bar{\varphi}$	Exact value of $E\bar{\varphi}$	$E\bar{\varphi}$	Exact value of $E\bar{\varphi}$
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
1	0.4717137	0.4717137	0.1410113	0.1410225	0.9999876	0.9999856
2	0.7725019	0.7725019	0.2877018	0.2877247	1.0000029	1.0000016
3	0.9307090	0.9307090	0.4465797	0.4466153	1.0000035	1.0000046
4	1.0138811	1.0138811	0.6260173	0.6260673	1.0000041	1.0000056
5	1.0601787	1.0601787	0.8377235	0.8377903	1.0000045	1.0000061
6	1.0874152	1,0874152	1.0987790	1.0988667	1.0000054	1.0000063
7	1.1039793	1.1039793	1.4332368	1.4333512	1.0000062	1.0000065
8	1.1140136	1.1140136	1.8655798	1.8657286	1.0000071	1.0000065
9	1.1196315	1.1196315	2.3704169	2.3706059	1.0000077	1.0000066
10	1.1218770	1.1218770	2.7244753	2.7246926	1.0000081	1.0000066
11	1.1211388	1.1211388	2.5914122	2.5916189	1.0000079	1.0000066
12	1.1172909	1.1172909	2.1160449	2.1162137	1.0000074	1.0000066
13	1.1096480	1.1096480	1.6363758	1.6365064	1.0000066	1.0000065
14	1.0967076	1.0967076	1.2550940	1.2551942	1.0000058	1.0000064
15	1.0755158	1.0755158	0.9607442	0.9608208	1.0000049	1.0000062
16	1.0402119	1.0402119	0.7269221	0.7269801	1.0000043	1.0000059
17	0.9786362	0.9786362	0.5330962	0.5331387	1.0000037	1.0000052
18	0.8645836	0.8645836	0.3651523	0.3651814	1.0000035	1.0000035
19	0.6446554	0.6446554	0.2132700	0.2132869	0.9999993	0.9999975
20	0.2518227	0.2518227	0.0701600	0.0701655	0.9999240	0.9999218

The solution of problem (1) with the data

$$f(s) = N_-\bar{\Phi} = -\pi a \Delta^{-1} U \cos \pi s \qquad (\boldsymbol{x} \in \partial\omega) \tag{14}$$

is sought in the form $\varphi(\lambda, \mu) = \Phi(\lambda, \mu) + Uz$, $\Phi(\lambda, \mu) = V[\chi](\lambda, \mu)$. Its exact solution is also written explicitly:

$$\chi(\mu) = -0.25a\Delta^{-1}\Lambda_1 U \cos \pi s, \qquad \bar{\varphi}(\lambda_0, \mu) = -b\Lambda_1 U \cos \pi s,$$

$$E\bar{\varphi}(\lambda_0, \mu) = \pi b \Delta^{-1} \Lambda_1 U \sin \pi s, \qquad \Lambda_1^{-1} = (\lambda_0^2 \mp 1) Q_1(\lambda_0) - 1. \tag{15}$$

In Eq. (15), we have

$$Q_1(\lambda_0) = \begin{cases} [\lambda_0 \ln((\lambda_0 + 1)/(\lambda_0 - 1)) - 2]/2, & b > a, \\ 1 - \lambda_0 \arcsin({\lambda_0}^2 + 1)^{-1/2}, & b < a. \end{cases}$$

The computed results are summarized in Tables 1 and 2.

Table 1 contains the numerical solutions of the exterior Neumann problem (2) with the right side (12) for the ellipsoids of revolution extended ($a/b = 1/100$) and flattened ($a/b = 100/1$) along the axis of revolution z. The solutions were constructed numerically by formulas (5) on the basis of the numerical solution of Eq. (3): the solution was calculated in the nodes s_j ($0 \leqslant j \leqslant 20$) by the formula $\bar{\Phi}(\lambda_0, \cos \pi s_j) = \bar{V}[\psi](s_j)$. The exact values of the solution $\bar{\Phi}(\lambda_0, \cos \pi s_j)$ of this problem were found by formulas (13).

Table 2 gives the results of the numerical solution of the classical problem of the flow around the ellipsoids of revolution extended ($b > a$) and flattened ($b < a$) along the axis of symmetry z. The cases $a/b = 1/3$ and $3/1$ correspond to solutions of standard domains commonly used for test computations; therefore, no comments are given here. Table 2 also contains the results of the numerical solution of the flow problem for an ellipsoid of revolution significantly extended ($a/b = 1/1000$) along the axis of symmetry z. The sixth column of Table 2 gives the numerical solution of problem (3) with data (14) for $U \equiv 1$, the direct values of the tangential derivative $E\bar{\varphi}(\lambda_0, \cos \pi s_j)$ being determined by formulas (6) and (10). The seventh column of Table 2 contains the exact values $E\bar{\varphi}(\lambda_0, \cos \pi s_j)$ of the tangential velocity of fluid particles at the boundary of the ellipsoid of revolution, which were calculated by formulas (15) for $U \equiv 1$. The computation results listed in the sixth column of Table 2 demonstrate most clearly the potential capabilities of unsaturated numerical methods in C^∞-smooth axisymmetric problems of the flow around various bodies: an extremely efficient numerical algorithm of solving problem (1) was obtained by using C^∞-smoothness and the harmonicity of its solution (see Theorems 1–4) to the greatest possible extent.

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